Relationship Between Moduli of Smoothness of Function in Two Different Spaces

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Abstract

In this paper we prove an improved version of P.L .UL'yanov inequality in terms of the fractional order modulus of smoothness of 2π periodic function in different spaces.

Keywords: Direct Theorem, fractional difference, $L_P$spaces.

الخلاصة

برهنا في هذا البحث احد أنواع متراجحةUlyano'v بدلالة مقياس النعومة ذي الرتبة الكسرية للدوال دورة 2π في فضائين مختلفين .

الكلمات المفتاحية: المبرهنة المباشرة،الفرق ذي الرتبة الكسرية،الفضاء L_P .

1- Introduction

In [UL'yanov1968, UL'yanov 1970] UL'yanov proved if $g(x) \in L_P, 1 \leq p < q < \infty$, for any $k \in (0,1]$ then

$$\omega_1(g,k)_q \leq c_1 \left\{ \int_0^k \left(s^{-\left(\frac{1}{p}-\frac{1}{q}\right)} \omega_1(g,s)_p \right)^q \frac{ds}{s} \right\}^{1/q} \quad (1\text{-}1)$$

where c_1 is a positive constant does not depend on g and k .

In [Kolyada 1988] Kolyada strength (1-1) and proved if $g(x) \in L_p, 1 < p < q < \infty$, then for any $k \in (0,1]$ we have

$$k^{1-\frac{1}{p}+\frac{1}{q}} \left\{ \int_k^1 \left(t^{\frac{1}{p}-\frac{1}{q}-1} \omega_1(g,t)_q \right)^p \frac{dt}{t} \right\}^{1/p} \leq c_2 \left\{ \int_0^k \left(t^{-\left(\frac{1}{p}-\frac{1}{q}\right)} \omega_1(g,t)_p \right)^q \frac{dt}{t} \right\}^{1/q},$$

where c_2 is a positive constant does not depend on g and k .

In this article we improve the above inequalities in terms of module of higher orders i.e. we prove these inequalities for $0 < p , q < 1$ .Now let us introduce some definitions that can be used in this paper.

Let $L_p[0,2\pi]$,$0 < p < 1$, be the space of periodic measurable function $g(x)$with period 2π, under the norm $\|g\|_p = \left(\int_0^{2\pi} |g(x)|^p \right)^{\frac{1}{p}} dx < \infty$.The βth - difference of $g(x)$ at the point x , $x \in R$ of fractional order β $(\beta > 0)$is given by

$$\Delta_t^{\beta} g(x) = \sum_{i=0}^{\infty} (-1)^i \binom{\beta}{i} f(x + (\beta - i)t) \; [Tikhonov, 2004]$$

The βth modulus of smoothness in terms of fractional order $(\beta > 0)$ is defined by

$$\omega_{\beta}(g,t)_p = \sup_{|h| \le t} \left\| \Delta_h^{\beta} g(x) \right\| \text{ [Butzer, Dyckhoff and Goerlich 1977]}$$

The K - functional is defined by

$$K\left(g, t^{\beta}, L_p, w_p^{\beta}\right) = \inf_{f \in w_p^{\beta}} \left(\|g - f\|_p + \left\| D_f^{\beta} \right\|_p \right) \text{[Tikhonov 2004]}$$

And $(a_{n\downarrow} 0), (n \uparrow \infty)$means a_n is decreasing sequence converge to 0 as n approach to ∞ .

2-The Auxiliary Results

In this section we introduce some results that we need to prove of our Theorem

Lemma 2.1 [Sharba 2012]

$w_{\beta}(g,\lambda)_p$is non decreasing non negative function of λ on $(0,\infty)$ with $\lim_{\lambda \to 0^+} \omega_{\beta}(g,\lambda)_p = 0$

Lemma 2.2 [Sharba 2012]

For any $g \in L_p[-1,1]$, $0 < p < 1$, we have

$$E_n(g)_p \le C(p)\omega_r(g, n^{-1})_p n > r$$

Lemma 2.3 [Sharba 2012]

For a function $f \in L_p[a,b]$, $0 < p < 1$ we have

$$\omega_{\beta}(f,t)_p \approx K\left(f, t^{\beta}, L_p, w_p^{\beta}\right)$$

Lemma 2.4 [Nikol'skii 1975]

Let $a_n \ge 0$, $0 < \alpha \le \beta < \infty$. Then

$$\left(\sum_{n=1}^{\infty} a_n^{\beta} \right)^{\frac{1}{\beta}} \le \left(\sum_{n=1}^{\infty} a_n^{\alpha} \right)^{\frac{1}{\alpha}}$$

Lemma 2.5

Let $g(x) \in L_p[0,2\pi]$, $0 < p < 1$ we have

$$E_{2^n}(g)_q \leq c\left(\sum_{m=n}^{\infty} 2^{mq\left(\frac{1}{p}-\frac{1}{q}\right)} E_{2^m}^{q}(g)_p\right)^{\frac{1}{q}},$$

where c is positive constant does not depend on $g\ and\ n$.

Proof:

Using the same way used in Theorem (1) of [Potapov,S**i**monovandTikhonov 2008]

Proposition 2.6

Let $f(x) \in L_p[0,2\pi], 0 < p < 1, \beta > 0. Then\ for\ any\ m \in N$

i) $E_m(f)_p \leq c_3\omega_\beta\left(f, \frac{1}{m}\right)_p$,

ii) $c_4(m^{-\beta}\left\|S_m^{(\beta)}(f,x)\right\|_p + E_m(f)_p) \leq \omega_\beta\left(f, \frac{1}{m}\right)_p \leq c_5(m^{-\beta}\left\|S_m^{(\beta)}(f,x)\right\|_p + E_m(f)_p)$,

iii) $c_6\left\|\sum_{n=1}^{m} n^\beta A_n(x)\right\|_p \leq \left\|S_m^{(\beta)}(f,x)\right\|_p \leq c_7\left\|\sum_{n=1}^{m} n^\beta A_n(x)\right\|_p$,

where c_3, c_4, c_5, c_6, c_7 are positive constants do not depend on $f and\ m$.

Proof (i): Follows directly from Lemma 2.2

Proof (ii): Follows directly from Lemma 2.3 and part (i)

Proof (iii): Follows directly using the hypothesis

$$S_m^{(\beta)}(x) \sim \sum_{n=0}^{\infty}(n+1)^{-\beta}\epsilon_n A_n(x)\ such\ that\ |\epsilon_n| \leq 1$$

$$c_8\left\|\sum_{n=1}^{m}(n+1)^{-\beta} \epsilon_n A_n(x)\right\|_p \leq \left\|S_m^{(\beta)}(f,x)\right\|_p \leq c_9\left\|\sum_{n=1}^{m}(n+1)^{-\beta} \epsilon_n A_n(x)\right\|_p$$

□

Proposition 2.7

Let $f(x) \in L_p[0,2\pi]\ 0 < p < 1, f(x) \sim \frac{a_\circ}{2} + \sum_{n=1}^{\infty} a_n \cos nx\ , whrera_n \downarrow 0\ (n \uparrow \infty). Then$

$$c_{10}\left(a_\circ^p+\sum_{n=1}^{\infty}a_n^p n^{p-2}\right)^{\frac{1}{p}} \le \|f\|_p \le c_{11}\left(a_\circ^p+\sum_{n=1}^{\infty}a_n^p n^{p-2}\right)^{\frac{1}{p}},$$

where the positive constants c_{10}, c_{11} do not depend on f.

Proof:

$$\|f\|_p=\left\|\frac{a_\circ}{2}+\sum_{n=1}^{\infty}a_n\cos nx\right\|_p$$

$$=\left(\int_0^{2\pi}\left|\frac{a_\circ}{2}+\sum_{n=1}^{\infty}a_n\cos nx\right|^p dx\right)^{\frac{1}{p}}$$

$$\le\left(\int_0^{2\pi}\left|\frac{a_\circ}{2}\right|^p+\sum_{n=1}^{\infty}|a_n|^p|\cos nx|^p dx\right)^{\frac{1}{p}}$$

$$\le c(p)\left(|a_\circ|^p+\sum_{n=1}^{\infty}|a_n|^p\, n^p\right)^{\frac{1}{p}}$$

$$\|f\|_p=\left(\int_0^{2\pi}\left|\frac{a_\circ}{2}+\sum_{n=1}^{\infty}a_n\cos nx\right|^p dx\right)^{\frac{1}{p}}$$

Since $cosnx$ is bounded on $[0,2\pi]$ and $a_n\to 0$ so it is also bounded and the sum of any two bounded function it is also bounded functions so that

$$\|f\|_p \ge C(p)\left(|a_o|^p+\sum_{n=1}^{\infty}|a_n|^p n^{p-2}\right)^{\frac{1}{p}} \qquad \square$$

3- The main result

In this section we introduce our main results.

Theorem 3.1

Let $g(x) \in L_p, 0 < p < q < 1\,, \beta > 0,\; g(x) \sim \frac{a_\circ}{2} + \sum_{n=1}^{\infty} a_n \cos nx\;, a_n n^{\alpha+\frac{1}{p}-\frac{1}{q}} \downarrow 0\,,$ $(n \uparrow \infty). Then\ for\ any\ k \in (0,1]\ we\ have:$

$$\omega_\beta(g,k)_q \le c_{12} \left\{ \left(\left(\int_0^k t^{-\frac{1}{p}+\frac{1}{q}} \omega_{\beta+\frac{1}{p}-\frac{1}{q}}(g,t)_p \right)^q \frac{dt}{t} \right\}^{\frac{1}{q}},$$

where c_{12} is a positive constant does not depend on $g\ and\ k$.

Proof:

Using Lemma 2-1 and applying Proposition 2.6 for any $n = 0,1,2,\cdots$ we get

$$\omega_\beta\left(g,\frac{1}{2^n}\right)_q \le c(q)\omega_\beta\left(g,\frac{1}{2^{n+1}}\right)_q \le c(q)\left(2^{-(n+1)\beta}\left\|\sum_{n=1}^{2^{n+1}}(n+1)^{-\beta}\epsilon_n A_n(x)\right\|_q + E_{2^{n+1}}(g)_q\right)$$

Using Proposition 2.7 and Lemma 2.4 we get

$$\left\|\sum_{n=1}^{2^{n+1}}(n+1)^{-\beta}\epsilon_n A_n(x)\right\|_q \le c(q)\left(\sum_{n=1}^{2^{n+1}} \in_n^q (n+1)^{-\beta q}(n+1)^{q-2}\right)^{\frac{1}{q}}$$

$$\left.\begin{aligned}
&\le c(q)\left(\textstyle\sum_{n=1}^{2^n} \in_{2^n}^q 2^{-(n+1)^{\beta q}} 2^{(n+1)q(1-\frac{1}{q})}\right)^{\frac{1}{q}} \\
&\le c(p)\left(\sum_{n=1}^{2^n} \in_{2^n}^p 2^{-(n+1)^{\beta p}} 2^{(n+1)p(1-\frac{1}{q})}\right)^{\frac{1}{p}} \\
&\le c(p)\left(\sum_{n=1}^{2^n} \in_n^p (n+1)^{p\left(\beta+\frac{1}{p}-\frac{1}{q}\right)}(n+1)^{p-2}\right)^{\frac{1}{p}} \\
&\le c(p)\left\|\sum_{n=1}^{2^n}(n+1)^{\beta+\frac{1}{p}-\frac{1}{q}} \in_n A_n(x)\right\|_p
\end{aligned}\right\}(3-1)$$

Combining estimates in Lemma (2.5) and (3-1) and applying Proposition 2.6 we obtain

$$\omega_\beta\left(g,\frac{1}{2^n}\right)_q \le c(p)\left(\sum_{n=0}^{2^n} 2^{(n+1)q\left(\frac{1}{p}-\frac{1}{q}\right)}\,\omega^q_{\beta_{+\frac{1}{p}-\frac{1}{q}}}\left(g,\frac{1}{2^{n+1}}\right)_p\right)^{\frac{1}{q}},$$

Using properties of integral we finally get:

$$\omega_\beta(g,k)_q \le c(p)\left\{\int_0^k \left(t^{-\left(\frac{1}{p}-\frac{1}{q}\right)}\omega_{\beta+\frac{1}{p}-\frac{1}{q}}(g,t)_p\right)^q \frac{dt}{t}\right\}^{\frac{1}{q}},$$

Thus the proof is complete.

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